

# EXAMPLES OF NON $d_\omega$ -EXACT LOCALLY CONFORMAL SYMPLECTIC FORMS

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ABSTRACT.

We exhibit two three-parameter families of locally conformal symplectic forms on the solvmanifold  $M_{n,k}$  considered in [1], and show, using the Hodge-de Rham theory for the Lichnerowicz cohomology that that they are not  $d_\omega$  exact, i.e. their Lichnerowicz classes are non-trivial (Theorem 1). This has several important geometric consequences ( corollary 2, 3). This also implies that the group of automorphisms of the corresponding locally conformal symplectic structures behaves much like the group of symplectic diffeomorphisms of compact symplectic manifolds. We initiate the classification of the local conformal symplectic forms in each 3-parameter family (Theorem 2, corollary 1). We also show that the first ( and ) third Lichnerowicz cohomology classes are non-zero (Theorem 3). We observe finally that the manifolds  $M_{n,k}$  carry several interesting foliations and Poisson structures.

## 1. Introduction

Using a closed 1-form  $\omega$  on a smooth manifold  $M$ , we deform the de Rham differential  $d$  on differential forms  $\Lambda^*(M)$  into the operator  $d_\omega$ :

$$d_\omega \theta = d\theta + \omega \wedge \theta \quad (1)$$

It is easy to see that  $(d_\omega)^2 = 0$ . Hence, we can define  $H_\omega^*(M)$  as the quotient of  $\text{Ker } d_\omega$  by the image of  $d_\omega$ , and call it the Lichnerowicz cohomology of  $M$  relative to the 1-form  $\omega$  [7]. One proves that  $H_\omega^*(M)$  is isomorphic to the cohomology  $H^*(M, \mathcal{F}_\omega)$  of  $M$  with values in the sheaf of local functions  $f$  on  $M$  such that  $d_\omega f = 0$ . In [5],  $H_\omega^*(M)$  was also characterised as the cohomology of conformally

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1991 *Mathematics Subject Classification.* 53C12; 53C15.

*Key words and phrases.* ACFM manifold, locally conformal symplectic form, locally conformal Kaehler form Lichnerowicz cohomology, Lee form, parallel Lee form, extended Lee homomorphism.

invariant forms on the minimum regular cover of  $M$  over which  $\omega$  pulls back to an exact form.

If the 1-form  $\omega$  is exact, then the Lichnerowicz cohomology is isomorphic to the de Rham cohomology  $H^*(M, \mathbb{R})$  of  $M$ . But if  $\omega$  is not exact, these two cohomologies are very different. For instance if  $M$  is a compact oriented  $n$  dimensional manifold, then  $H_\omega^0(M) \approx H_\omega^n(M) = 0$  for any non-exact 1-form  $\omega$  [7], [8], [5].

There are few instances in which these cohomology groups are explicitly calculated. The goal of this paper is to compute some Lichnerowicz cohomology classes of the locally conformal symplectic manifold constructed by L.C.D. Andres, L.A. Cordero, M. Fernandez, and J.J. Mencia [1], herein nicknamed the "ACFM manifold".

A **locally conformal symplectic form** on a smooth manifold of dimension at least 4, is a non-degenerate 2-form  $\Omega$  such that there exists a closed 1-form  $\omega$  satisfying:

$$d\Omega = -\omega \wedge \Omega \quad (2)$$

The closed 1-form  $\omega$  above is uniquely determined by  $\Omega$  and is called the Lee form of  $\Omega$  [9]. The uniqueness of the Lee form is a consequence of the following elementary fact [5]

**Lemma 0.** *If a 2-form  $\Theta$  has rank at least 4 at every point, and  $\alpha$  is any 1-form, then  $\alpha \wedge \Theta = 0$  implies that  $\alpha$  is identically zero.*

If  $\Omega$  is a locally conformal symplectic form with Lee form  $\omega$ , then equation (2) says that  $d_\omega \Omega = 0$ . Hence  $\Omega$  represents an element  $[\Omega] \in H_\omega^2(M)$ .

All the examples known to the author so far were locally ( non global) conformal

symplectic forms  $\Omega$  with trivial Lichnerowicz class  $[\Omega]$  [5], [6], [8], [13].

We exhibit here examples of locally (non global) conformal symplectic forms with non trivial Lichnerowicz classes.

## 2. The ACFM manifold [1] and statement of the results

Let  $G_k$  be the group of matrices of the form:

$$A = \begin{pmatrix} e^{kz} & 0 & 0 & x \\ 0 & e^{-kz} & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where  $x, y, z \in \mathbb{R}$  and  $k \in \mathbb{R}$  is such that  $e^k + e^{-k} \in \mathbb{Z} - \{2\}$ . The group  $G_k$  is a connected solvable Lie group with coefficients:  $x(A) = x$ ,  $y(A) = y$ ,  $z(A) = z$  and a basis of right invariant 1-forms:

$$dx - kxdz, \quad dy + kydz, \quad dz$$

There exists a discrete subgroup  $\Gamma_k \subset G_k$  such that  $N_k = G_k/\Gamma_k$  is compact [2].

The basis (3) descends to a basis of 1-forms:

$$\alpha, \beta, \gamma \tag{3}$$

on  $N_k$ .

The forms  $\gamma$  and  $\alpha \wedge \beta$  are closed and their cohomology classes  $[\gamma]$ ,  $[\alpha \wedge \beta]$  generate  $H^1(M, \mathbb{R})$  and  $H^2(M, \mathbb{R})$  respectively.

Let  $\lambda \in \mathbb{R}$  be a number such that  $\lambda[\alpha \wedge \beta] \in H^2(M, \mathbb{Z})$ .

### Definition.

*For any non zero  $n \in \mathbb{Z}$ , the ACFM manifolds  $M_{k,n}$  are the total spaces of principal  $S^1$  bundles over  $N_k$  with Chern class  $n\lambda[\alpha \wedge \beta]$ .*

For simplicity, we will denote the pull back to  $M_{k,n}$  (by the projection  $M_{k,n} \rightarrow N_k$ ) of a form  $\theta$  on  $N_k$ , again by  $\theta$ .

If  $\eta$  is a connection form on  $M_{k,n}$  with curvature  $n\lambda(\alpha \wedge \beta)$ , then:

$$d\eta = n\lambda(\alpha \wedge \beta). \quad (4)$$

The set

$$\{\alpha, \beta, \gamma, \eta\} \quad (5)$$

form a basis of 1-forms on the 4-dimensional manifold  $M_{k,n}$ .

Our main results are contained in theorems 1, 2, 3 below:

**Theorem 1.**

(i) For each real numbers  $(t_1, t_2, t_3)$ , with  $t_1.t_2 \neq 0$  and  $t_1 t_2 \neq nk\lambda t_3^2$  the 3-parameter family of 2-forms

$$\Omega_{(t_1, t_2, t_3)} = t_1(\alpha \wedge \eta) + t_2(\beta \wedge \gamma) + t_3(n\lambda(\alpha \wedge \beta) - k\gamma \wedge \eta)$$

are locally conformal symplectic forms, with the same Lee form  $\omega = -k\gamma$ , and their Lichnerowicz classes  $[\Omega_{(t_1, t_2, t_3)}] \in H_{\omega}^2(M_{n,k})$  are non-trivial.

(ii) For each real numbers  $(s_1, s_2, s_3)$ , with  $s_1.s_2 \neq 0$  and  $s_1 s_2 \neq n\lambda k s_3^2$ , the 3-parameter family of 2-forms

$$\Omega_{(s_1, s_2, s_3)} = s_1(\beta \wedge \eta) + s_2(\alpha \wedge \gamma) + s_3(n\lambda(\alpha \wedge \beta) + k\gamma \wedge \eta)$$

are locally conformal symplectic forms, with the same Lee form  $\omega' = k\gamma$ , and their Lichnerowicz classes  $[\Omega'_{(s_1, s_2, s_3)}] \in H_{(-\omega)}^2(M_{n,k})$  are non-trivial.

**Remark 1**

The local conformal symplectic forms above are not global conformal symplectic since their Lee form is not exact [5]. Besides,  $M_{n,k}$  can not carry a symplectic form since  $H^2(M_{n,k}, \mathbb{R}) = 0$  [1].

**Remark 2**

The 2-forms

$$\Omega = n\lambda(\alpha \wedge \beta) - k\gamma \wedge \eta = d_\omega(\eta)$$

$$\Omega' = n\lambda(\alpha \wedge \beta) + k\gamma \wedge \eta = d_{\omega'}(\eta)$$

are exact locally conformal symplectic forms. They are obviously non-degenerate and  $d_\omega \Omega = (d_\omega)^2 \eta = 0$ . Same thing with  $\Omega'$ . Since the forms in theorem 1 are not exact, we have the following

**Corollary 1.**

*The locally conformal symplectic form  $\Omega$ , resp  $\Omega'$  is not equivalent to  $\Omega_{(t_1, t_2, t_3)}$ , resp.  $\Omega_{(s_1, s_2, s_3)}$ , i.e. there are no diffeomorphisms  $\phi$  of  $M_{n,k}$  and smooth function  $f$  such that  $\phi^* \Omega = f \Omega_{(t_1, t_2, t_3)}$  ( similarly for  $\Omega'$  and  $\Omega_{(s_1, s_2, s_3)}$ ),*

**Corollary 2.**

*No Lie group can act transitively on  $M_{n,k}$  preserving  $\Omega_{(t-1, t_2, t_3)}$  or  $\Omega_{(s_1, s_2, s_3)}$ .*

**Theorem 2.**

*If  $(t_1, t_2, t_3) \in \mathbb{R}^3$  satisfy  $t_2 = e^u$  and  $t_1 \cdot e^{ut} \neq nk\lambda st_3^2$  for all  $t, s \in [0, 1]$ , there exists a family of diffeomorphisms  $\phi_t$  with  $\phi_0 = id$  and*

$$\phi_t^*(\Omega_{(t_1, t_2, t_3)}) = f_t((t_1(\alpha \wedge \eta) + \beta \wedge \gamma).$$

*We have a similar statement for  $\Omega_{(s_1, s_2, s_3)}$ .*

**Question 1**

Consider the open subset  $U \subset \mathbb{R}^3$  defined by  $U = \{(t_1, t_2, t_3) | t_1 t_2 \neq nk\lambda t_3^2\}$ . The hypothesis in theorem 2 means that the point  $p = (t_1, t_2, t_3) \in \mathbb{R}^3$  and the point  $q = (t_1, 1, 0)$  belong to the same path component of  $U$ .

If  $p, q \in U$  belong to two different path components, are  $\Omega_{(p)}$  and  $\Omega_{(q)}$  still equivalent?

Here we can not use Moser's theorem, because we do not have a path of locally conformal symplectic forms. The path degenerates when we go from one component of  $U$  to another.

For instance, take  $t_1 = n\lambda/k$ ,  $t_2 = 1$ , then  $t_3$  must be different from  $-1/k$  and  $+1/k$ . All the local conformal symplectic forms  $\Omega_{(n\lambda/k, 1, t_3)}$  are equivalent for  $t_3$  in the interval  $[(-1/k) + \epsilon, (1/k) - \epsilon]$ , and  $\epsilon$  a small positive number.

Are the forms  $\Omega_{(n\lambda/k, 1, 0)}$  and  $\Omega_{(n\lambda/k, 1, (k+1)/k)}$  equivalent?

**Remark 3**

The locally conformal symplectic form  $\Omega = (n\lambda/k)(\alpha \wedge \eta) + \beta \wedge \gamma$  was found in [1] where the authors showed that it is a locally conformal Kaehler form, with non parallel Lee form, with respect to the metric  $g = \alpha^2 + \beta^2 + \gamma^2 + \eta^2$ .

In [10], it is proved that if a compact manifold  $M$  has a riemannian metric with respect to which a nowhere zero closed 1-form  $\omega$  is parallel, then  $H_\omega^*(M) = 0$ . This result and theorem 1 imply a stronger result:

**Corollary 3.**

*There is no riemannian metric on  $M_{n,k}$  with respect to which  $\pm k\gamma$  is parallel.*

Vaisman and Goldberg [15] have found sufficient conditions for the Lee form of

a compact locally Kaehler manifold to be parallel:

### Theorem

*Let  $(M, J, g)$  be a compact locally Kaehler manifold with Lee form  $\omega$  a nowhere vanishing form. If the Ricci tensor of  $(M, g)$  is positive semi-definite and vanishes only in the direction of  $B$ , where  $B$  is the vector field defined by  $i(B)\Omega = \omega$ , and  $\Omega$  is the locally conformal Kaehler form, then  $\omega$  is parallel.*

### Question 2

Let  $\Omega$  be the locally conformal Kaehler form in the Goldman-Vaisman theorem above, then  $H_\omega^*(M) = 0$ . In particular  $\Omega$  is  $d_\omega$ -exact.

Is Goldberg-Vaisman theorem above still true for locally conformal symplectic manifolds?

This would give a sufficient condition for the vanishing of the Lichnerowicz cohomology and in particular the  $d_\omega$ -exactness of the locally conformal symplectic form.

### Theorem 3.

*The forms  $\alpha$ ,  $\alpha \wedge \eta$  and  $\alpha \wedge \gamma \wedge \eta$  represent non zero elements in  $H_\omega^i(M_{n,k})$ ,  $i = 1, 2, 3$ .*

*The forms  $\beta$ ,  $\beta \wedge \eta$ ,  $\beta \wedge \gamma \wedge \eta$  represent non zero elements in  $H_{(-\omega)}^i(M_{n,k})$ ,  $i = 1, 2, 3$ .*

*The classes  $[\Omega_{(t_1, t_2, t_3)}]$  in theorem 1 coincide with  $t_1[\alpha \wedge \eta]$ . Similarly,  $[\Omega_{(s_1, s_2, s_3)}] = s_1[\beta \wedge \eta]$ .*

### Question 3

Theorem 3 says that the dimension of  $H_{\{\omega, -\omega\}}^i(M_{n,k})$ , for  $i = 1, 2, 3$  is at least 1.

Is it exactly one ?

### Remarks on the group of automorphisms and the Lie algebra of infinitesimal automorphisms

Two locally conformal symplectic forms  $\Omega$  and  $\Omega'$  are said to be equivalent if there exists a no-where zero function  $f$  such that  $\Omega' = f\Omega$ .

A locally conformal symplectic structure  $\mathcal{S}$  is an equivalence class of locally conformal symplectic forms. Let  $\mathcal{S}_t$ , respectively  $\mathcal{S}_s$  be the locally conformal symplectic structures on  $M_{n,k}$  represented by the forms  $\Omega_{(t_1, t_2, t_3)}$  respectively  $\Omega_{(s_1, s_2, s_3)}$ , and let  $G(\mathcal{S}_t)$ ,  $G(\mathcal{S}_s)$  be the corresponding automorphism groups, i.e. the group of all diffeomorphisms  $\phi$  of  $M_{n,k}$  such that  $\phi^*(\Omega_{(t_1, t_2, t_3)}) = f\Omega_{(t_1, t_2, t_3)}$  for some no-where zero function  $f$  (same definition for  $\Omega_{(s_1, s_2, s_3)}$ ).

Let  $\mathcal{L}(\mathcal{S}_t(M_{n,k}))$  and  $\mathcal{L}(\mathcal{S}_s(M_{n,k}))$  be the Lie algebra of infinitesimal automorphisms of the locally conformal symplectic structures  $\mathcal{S}_t$  and  $\mathcal{S}_s$ . This is the Lie algebra of vector fields  $X$  on  $M_{n,k}$  such that  $L_X\Omega_{(t_1, t_2, t_3)} = f\Omega_{(t_1, t_2, t_3)}$ , respectively  $L_X\Omega_{(s_1, s_2, s_3)} = f\Omega_{(s_1, s_2, s_3)}$ , for some function  $f$ , where  $L_X$  stands for the Lie derivative in the direction  $X$ .

Let  $M$  be a compact manifold equipped with a locally conformal symplectic form  $\Omega$ , with Lee form  $\omega$  and such that  $[\Omega]$  is a non trivial element of  $H_\omega^2(M)$ , then the groups of automorphisms of the corresponding locally conformal symplectic structure, and the corresponding Lie algebra, behave very much like in the symplectic case [8]. For instance there is a Calabi invariant on the identity component of the group with values in a quotient of  $H_\omega^1(M)$  and its kernel is a simple group [8], exactly like in the symplectic case [3]. At the level of the Lie algebras, the mapping



$X \mapsto i(X)\Omega$  is a surjective Lie algebra homomorphism into  $H_\omega^1(M)$  whose kernel is the commutator subalgebra [8]. The fact that  $i(X)\Omega$  is  $d_\omega$  closed comes from the following fact:

**Corollary 4.**

*For every vector field  $X \in \mathcal{L}(\mathcal{S}_t(M_{n,k}))$  , then*

$$L_X \Omega_{(t_1, t_2, t_3)} = k\gamma(X)\Omega_{(t_1, t_2, t_3)}.$$

*For  $X \in \mathcal{L}(\mathcal{S}_s(M_{n,k}))$  , then*

$$L_X \Omega_{(s_1, s_2, s_3)} = -k\gamma(X)\Omega_{(s_1, s_2, s_3)}.$$

An immediate consequence of corollary 4 is the following fact: for any infinitesimal automorphism  $X \in \mathcal{L}(\mathcal{S}_t(M_{n,k}))$  or in  $X \in \mathcal{L}(\mathcal{S}_s(M_{n,k}))$  :

$$\int_{M_{n,k}} \gamma(X)(\Omega_{(t_1, t_2, t_3)})^2 = \int_{M_{n,k}} \gamma(X)(\Omega_{(s_1, s_2, s_3)})^2 = 0.$$

### 3. Proofs of the results

The proofs rest on the Hodge-de Rham theory for the  $d_\omega$  operator and lemma 1 below.

#### The Hodge-de Rham theory for the $d_\omega$ operator [7]

Let  $(M, g)$  be a compact oriented  $n$ -dimensional riemannian manifold and  $\omega$  a 1-form on  $M$ . Let  $*$  be the Hodge-de Rham star operator defined by  $g$  and  $\delta$  the codifferential:  $\delta(\theta) = (-1)^{(nl+n+1)}(* \circ d \circ *)\theta$  for  $\theta \in \Lambda^l(M)$ , and let  $\mathcal{U}_\omega$  be the operator:  $\mathcal{U}_\omega(\theta) = (-1)^{nl+n}(*(\omega \wedge *\theta))$  for  $\theta \in \Lambda^l(M)$ .

We let

$$\delta_\omega = \delta + \mathcal{U}_\omega$$

Consider the inner product  $\langle \cdot, \cdot \rangle$  on  $\Lambda^l(M)$ :

$$\langle \rho, \nu \rangle = \int_M \rho \wedge * \nu.$$

One has that  $\langle d_\omega \rho, \nu \rangle = \langle \rho, \delta_\omega \nu \rangle$ . Since  $M$  is compact and  $(\Lambda^*(M), d_\omega)$  is elliptic, we get the following result, proved in [7]:

**Theorem.**

*We have the orthogonal decomposition:*

$$\Lambda^l(M) = (\mathcal{H}_\omega^l(M)) \oplus (d_\omega(\Lambda^{l-1}(M))) \oplus (\delta_\omega(\Lambda^{l+1}(M)))$$

where

$$\mathcal{H}_\omega^l(M) = \{\theta \in \Lambda^l \mid d_\omega \theta = \delta_\omega \theta = 0\}$$

is the space of  $\omega$ -harmonic forms.

Finally, we have:

$$H_\omega^l(M) \approx \mathcal{H}_\omega^l(M).$$

On the manifold  $M_{n,k}$ , we will put the riemannian metric  $g$  which makes the forms  $\alpha, \beta, \gamma, \eta$  orthonormal, i.e.  $g = \alpha^2 + \beta^2 + \gamma^2 + \eta^2$ . We will consider the closed 1-forms  $\omega = -k\gamma$  and  $(-\omega)$  respectively.

**Lemma 1.**

*If  $\alpha, \beta, \gamma, \eta$  is the basis (5) of 1-forms on  $M_{k,n}$ , we have:*

$$d\alpha = -k\alpha \wedge \gamma \tag{6}$$

$$d\beta = k\beta \wedge \gamma \tag{7}$$

*Proof.*

Let  $\{X, Y, Z, T\}$  be the basis of global vector fields on  $M_{k,n}$  dual to the basis of 1-form (5). A general 2-form  $\theta$  on  $M_{k,n}$  is uniquely written as :

$$\theta = A(\alpha \wedge \beta) + B(\alpha \wedge \gamma) + C(\alpha \wedge \eta) + D(\beta \wedge \gamma) + E(\beta \wedge \eta) + F(\gamma \wedge \eta),$$

where

$$A = \theta(X, Y), \quad B = \theta(X, Z),$$

$$C = \theta(X, T), \quad D = \theta(Y, Z),$$

$$E = \theta(Y, T), \quad F = \theta(Z, T).$$

If  $\rho$  denotes any of the basic 1-forms  $\alpha, \beta, \gamma, \eta$  and  $\xi, \xi'$  any pair of the vector fields  $\{X, Y, Z, T\}$  dual to the basis of 1-forms above, then:

$$(d\rho)(\xi, \xi') = -\rho([\xi, \xi']) + \xi \cdot \rho(\xi') - \xi' \cdot \rho(\xi) = -\rho([\xi, \xi']) \quad (8)$$

since  $\rho(\xi), \rho(\xi')$  are 1 or 0, and hence their directional derivatives are zero.

We have the following facts [1]:

$$[X, Z] = kX$$

$$[X, Y] = -n\lambda T$$

$$[Y, Z] = -kY$$

$$[X, T] = [Y, T] = [Z, T] = 0$$

Applying the above facts to  $\theta = d\alpha$ , and using the commutation relations above, we see that the only non zero coefficient is  $B = -k$ . Hence

$$d\alpha = -k\alpha \wedge \gamma$$

A similar inspection for  $d\beta$  yields

$$d\beta = k\beta \wedge \gamma$$

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□

**Proposition 1.**

1. Let  $\omega = -k\gamma$ , then  $\alpha$ ,  $\alpha \wedge \eta$ ,  $\alpha \wedge \gamma \wedge \eta$  are  $\omega$ -harmonic.

The form  $\beta \wedge \gamma$  is  $d_\omega$ -exact. In fact,  $d_\omega((1/2k)\beta) = \beta \wedge \gamma$ .

2. Let  $\omega' = k\gamma$ , then  $\beta$ ,  $\beta \wedge \eta$ ,  $\beta \wedge \gamma \wedge \eta$  are  $\omega'$ -harmonic.

The form  $\alpha \wedge \gamma$  is  $d_{\omega'}$ -exact. In fact,  $d_{(-\omega)}((-1/2k)\alpha) = \alpha \wedge \gamma$ .

*Proof.*

By lemma 1,  $d\alpha = -k\alpha \wedge \gamma = -\omega \wedge \alpha$ . Hence  $d_\omega \alpha = 0$ . Using (4), we get:

$$d(\alpha \wedge \eta) = -k\alpha \wedge \gamma \wedge \eta - \alpha \wedge (n\lambda\alpha \wedge \beta) = -k\alpha \wedge \gamma \wedge \eta = -\omega \wedge (\alpha \wedge \eta). \text{ Therefore } d_\omega(\alpha \wedge \eta) = 0.$$

$$d(\alpha \wedge \gamma \wedge \eta) = -k\alpha \wedge \gamma \wedge \gamma \wedge \eta - \alpha \wedge \gamma \wedge (n\lambda\alpha \wedge \beta) = 0 \text{ and } \omega \wedge (\alpha \wedge \gamma \wedge \eta) = 0.$$

Hence  $d_\omega(\alpha \wedge \gamma \wedge \eta) = 0$ .

We just have proved that the forms listed in (1) are  $d_\omega$  closed. Let us now show that  $\delta_\omega$  of these forms are zero:

$$\text{Up to a sign, } *\alpha = \beta \wedge \gamma \wedge \eta, \text{ hence } d*\alpha = k\beta \wedge \gamma \wedge \gamma \wedge \eta + \beta \wedge \gamma \wedge (n\lambda\alpha \wedge \beta) = 0.$$

Hence  $\delta(\alpha) = 0$ . On the other hands,  $\omega \wedge *\alpha = -k\gamma \wedge \beta \wedge \gamma \wedge \eta = 0$ , i.e.  $\mathcal{U}_\omega(\alpha) = 0$ , and hence  $\delta_\omega \alpha = 0$ .

Up to a sign,  $*(\alpha \wedge \eta) = \beta \wedge \gamma$ , so  $d*(\alpha \wedge \eta) = k\beta \wedge \gamma \wedge \gamma = 0$ . This Shows that  $\delta(\alpha \wedge \eta) = 0$ .

We have that  $\omega*(\alpha \wedge \eta) = -k\gamma \wedge \beta \wedge \gamma = 0$ . Hence  $\mathcal{U}_\omega = 0$ , and thus  $\delta_\omega(\alpha \wedge \eta) = 0$ .

Up to a sign,  $*(\alpha \wedge \gamma \wedge \eta) = \beta$ . Hence  $*d*(\alpha \wedge \gamma \wedge \eta) = *(k\beta \wedge \gamma) = k\alpha \wedge \eta$ .

Hence  $\delta(\alpha \wedge \gamma \wedge \eta) = (-1)^{17}k\alpha \wedge \eta = -k\alpha \wedge \eta$ .

On the other hands  $\mathcal{U}_\omega(\alpha \wedge \gamma \wedge \eta) = + *((-k\gamma) \wedge *(\alpha \wedge \gamma \wedge \eta)) = *(-k\gamma \wedge \beta) = *(k\beta \wedge \gamma) = k\alpha \wedge \eta$ . Hence  $\delta_\omega(\alpha \wedge \gamma \wedge \eta) = 0$ .

Finally,  $d_\omega\beta = k\beta \wedge \gamma + (-k\gamma) \wedge \beta = 2k(\beta \wedge \gamma)$ . Therefore  $\beta \wedge \gamma = d_\omega((1/2k)\beta)$ .

This proves (1). Similar calculations using  $\omega' = +k\gamma$  yields (2).  $\square$

### End of the proof of theorem 1

Observe first that

$$n\lambda\alpha \wedge \beta - k\gamma \wedge \eta = d\eta + \omega \wedge \eta = d_\omega\eta$$

( $\omega = -k\gamma$ ). Therefore:

$$\Omega_{(t_1, t_2, t_3)} = t_1(\alpha \wedge \eta) + d_\omega((t_2/2k)\beta + t_3\eta) \quad (9)$$

Since  $d_\omega(\alpha \wedge \eta) = 0$  and  $(d_\omega)^2 = 0$ , it follows that  $\Omega_{(t_1, t_2, t_3)}$  is  $d_\omega$ -closed.

Similarly,

$$\Omega_{(s_1, s_2, s_3)} = s_1(\beta \wedge \eta) + d_{\omega'}((-s_2/2k)\alpha + s_3\eta). \quad (10)$$

hence  $\Omega_{(s_1, s_2, s_3)}$  is  $d_{\omega'}$ -closed ( with  $\omega' = k\gamma$ ).

The Lichnerowicz cohomology classes of these forms  $[\Omega_{(t_1, t_2, t_3)}]$  are  $t_1[\alpha \wedge \eta]$  and  $[\Omega_{(s_1, s_2, s_3)}] = s_1[\beta \wedge \eta]$ , which are non-trivial, since  $\alpha \wedge \eta$  and  $\beta \wedge \eta$  are non-zero harmonic forms by proposition 2. By the Hodge-de Rham theory, they represent non trivial elements of  $H_\omega^2(M_{n,k})$ , respectively  $H_{(-\omega)}^2(M_{n,k})$ .

The forms in (i) and (ii) are non-degenerate. Indeed an immediate calculation gives:

$$(\Omega_{(t_1, t_2, t_3)})^2 = 2(t_1t_2 - nk\lambda t_3^2)(\alpha \wedge \beta \wedge \gamma \wedge \eta) \neq 0 \text{ iff } t_1t_2 - nk\lambda t_3^2 \neq 0$$

$(\Omega_{(s_1, s_2, s_3)})^2 = -2(s_1 s_2 - nk\lambda s_3^2)(\alpha \wedge \beta \wedge \gamma \wedge \eta) \neq 0$  iff  $s_1 s_2 - nk\lambda s_3^2 \neq 0$ . This completes the proof of theorem 1.  $\square$

### Proof of theorem 3

Proposition 1 shows that the forms considered in theorem 3 are  $\omega$  or  $\omega'$  harmonic. Hence they represent non-zero elements in the corresponding Lichnerowicz cohomologies.  $\square$

### Proof of theorem 2

We will use the locally conformal symplectic forms version of Moser theorem [11] proved in [5]:

#### Theorem.

*Let  $\Omega_t$  be a smooth family of locally conformal symplectic forms on a compact manifold  $M$ . Suppose that for all  $t$ , the Lee form of  $\Omega_t$  is the same 1-form  $\omega$  and  $\Omega_t - \Omega_0$  is  $d_\omega$  exact, then there exists a smooth family of diffeomorphisms  $\phi_t$ , with  $\phi_0 = id$  and a smooth family of functions  $f_t$  such that  $\phi_t^* \omega_t = f_t \Omega_0$*

Let  $(t_1, t_2, t_3) \in \mathbb{R}^3$  such that  $t_2 = e^u$  and  $t_1 \cdot e^{ut} \neq nk\lambda s t_3$  for all  $t, s \in [0, 1]$ , then

$$\Omega_{(t_1, t_3)}^{(s, t)} = t_1(\alpha \wedge \eta) + e^{ut}(\beta \wedge \gamma + e^{-ut} s t_3 d_\omega \eta) = t_1 \alpha \wedge \eta + d_\omega(e^{ut}((1/2k)\beta) + e^{-ut} s t_3 \eta))$$

is a smooth 2-parameter family of locally conformal symplectic forms, all with the same cohomology class  $t_1[\alpha \wedge \eta]$ , with the same Lee form  $\omega = -k\gamma$ .

By Moser theorem for locally conformal symplectic forms, for fixed  $s$ , there exists a smooth family of diffeomorphisms  $\psi^t$  depending smoothly on the parameter  $s$ , such that  $\psi^0 = id$  and  $(\psi^t)^* \Omega_{(t_1, t_3)}^{(s, t)} = f^{(t, s)} \Omega_{(t_1, t_3)}^{(0, s)}$ , for some functions  $f^{(t, s)}$ ,

depending smoothly on  $s$  and  $t$ .

Apply again Moser theorem to the family  $\Omega_s = \Omega_{(t_1, t_3)}^{(0, s)}$ . There is a family of diffeomorphisms  $\psi'_s$  such that  $(\psi'_s)^* \Omega_s = g_s \Omega_0 = g_s((t_1 \alpha \wedge \eta) + \beta \wedge \gamma)$ .

The required family of diffeomorphism is  $\psi^t \circ \psi'_s$  : it pulls  $\Omega_{(t_1, t_3)}^{(s, t)}$  back to a multiple by a function of  $\Omega_{(t_1, t_3)}^{(0, 0)} = t_1(\alpha \wedge \eta + \beta \wedge \gamma)$ .  $\square$

### Proof of corollary 3

Since  $\Omega_{(t_1, t_2, t_3)}$  resp.  $\Omega_{(s_1, s_2, s_3)}$  are non global and not  $\omega$ -exact, resp.  $\omega'$ -exact, proposition 2.4 of [13] implies that  $M_{n, k}$ , equipped with the forms above, is not a homogeneous locally conformal symplectic manifold.  $\square$

### Proof of corollary 4

We need to recall the generalized Lee homomorphism [13], [6].

Let  $\Theta$  be a 2-form of rank everywhere larger or equal to 4, such that there exists a closed 1-form  $\eta$  satisfying

$$d\Theta = -\eta \wedge \Theta$$

on a smooth manifold  $M$ . We know that  $\eta$  is uniquely determined by  $\Theta$  ( lemma 0).

Let  $\mathcal{L}(\Theta, M)$  denote the Lie algebra of vector fields  $X$  on  $M$  such that  $L_X \Theta = \mu_X \Theta$ .

For  $X \in \mathcal{L}(\Theta, M)$ , set  $\theta = i(X)\Theta$ , where  $i(\cdot)$  is the interior product operator.

We have

$$\begin{aligned} L_X \Theta &= \mu_X \Theta = di(X)\Theta + i(X)d\Theta \\ &= d\theta + i(X)(-\eta \wedge \Theta) \\ &= d\theta - \eta(X)\Theta + \eta \wedge i(X)\Theta \end{aligned}$$

$$= d\theta + \eta \wedge \theta - \eta(X)\Theta = d_\eta\theta - \eta(X)\Theta$$

Hence:

$$d_\eta\theta = l(X)\Theta \tag{11}$$

where

$$l(X) = \mu_X + \eta(X). \tag{12}$$

**Proposition 2.**

*If the manifold  $M$  is connected, then the function  $l(X)$  is a constant for all  $X \in \mathcal{L}(\Theta, M)$ , and the map  $l : \mathcal{L}(\Theta, M) \rightarrow \mathbb{R}$ ,  $X \mapsto l(X)$ , is a Lie algebra homomorphism*

A simple proof that  $l(X)$  is constant goes as follows:

$$\begin{aligned} 0 &= (d_\eta)^2\theta = d_\eta(l(X)\Theta) = d(l(X)\Theta) + \eta \wedge (l(X)\Theta) \\ &= d(l(X)) \wedge \Theta + l(X)(-\eta \wedge \Theta) + \eta \wedge (l(X)\Theta) \\ &= d(l(X)) \wedge \Theta \end{aligned}$$

By lemma 0,  $d(l(X)) = 0$ . Therefore  $l(X)$  is a constant, since  $M$  is connected.

This homomorphism was constructed by Vaisman [13] in the case where  $\theta$  is a locally conformal symplectic form.

As an immediate consequence of proposition 2 we get the following proposition, which belongs to the folklore:

**Proposition 3.**

*A locally conformal symplectic form  $\Omega$  with Lee form  $\omega$  on a connected manifold is  $d_\omega$ -exact iff there exists an infinitesimal automorphism  $X$  such that  $l(X) \neq 0$ .*



For the convenience of the reader, we include a proof.

*Proof.*

If  $X$  is an infinitesimal automorphism such that  $l(X) \neq 0$ , set :  $\theta = (1/l(X))i(X)\Omega$ , then (11) yields :  $\Omega = d_\omega\theta$ .

Converseky, if  $\Omega = d_\omega\theta$  for some 1-form  $\theta$ . Define a vector field  $X$  by the equation  $i(X)\Omega = \theta$ . An easy calculation shows that  $L_X\Omega = (1 - \omega(X))\Omega$ . This proves that  $X$  is an infinitesimal automorphism:  $L_X\Omega = \mu_X\Omega$  with  $\mu_X = 1 - \omega(X)$ , that is  $l(X) = 1$ .

□

The proof of corollary 2 is an immediate consequence of theorem 1. The Lee homomorphism corresponding to our locally conformal symplectic forms is trivial. Hence if a vector field  $X$  is such that  $L_X\Omega_{(t_1, t_2, t_3)} = f\Omega_{(t_1, t_2, t_3)}$ , then (12) yields :  $f = -\omega(X)$ .

□

#### 4. Some foliations and Poisson structures on $M_{n,k}$

The manifold  $M_{n,k}$  carries several remarkable foliations:

The codimension 1 foliations  $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ , defined by the integrable forms  $\gamma, \alpha$  and  $\beta$ .

The five 2-dimensional foliations  $\mathcal{F}_4, \dots, \mathcal{F}_8$ , tangent to the involutive distributions

$$\{X, Y\}, \{Y, Z\}, \{X, T\}, \{Y, T\}, \{Z, T\}$$

For instance the tangent space to the leaves of the foliation  $\mathcal{F}_1$  defined by  $\gamma = 0$ , is spanned by  $X, Y, T$ , The restriction of the locally conformal symplectic form  $\Omega = \alpha \wedge \eta + \beta \wedge \gamma$  is the 2-form  $\alpha \wedge \eta$ , whose kernel is spanned by  $Y$ . The quotient of

each leaf by the trajectories of  $Y$  is spanned by  $X, T$ , which define the 2-dimensional foliation  $\mathcal{F}_7$ . The 2-form  $\alpha \wedge \eta$  is a symplectic form on each leaf. We thus see that  $\mathcal{F}_7$  is a symplectic foliation.

The restriction of  $\eta$  to the leaves of the foliation  $\mathcal{F}_1$  is a contact form since  $\eta \wedge \eta = n\lambda\alpha \wedge \beta \wedge \eta$ . This is why we have the locally conformal symplectic forms  $\Omega$  and  $\Omega'$  in remark 2.

On each leaf of the 2 dimensional foliations there are is an obvious symplectic forms. Hence they are all symplectic foliations.

According to Vaisman [14], a symplectic foliation gives rise to a Poisson structure. The bracket of two functions is the so-called Dirac bracket: let  $\mathcal{F}$  be a symplectic foliation on a manifold  $M$ , the Dirac bracket  $\{f, g\}$  of two functions  $f, g$  on  $M$  is  $\{f, g\}(x) = (\Omega_F)(x)(X_{(f|_F)}, X_{(g|_F)})(x)$ , where  $\Omega_F$  is the symplectic form on the leaf  $F$  through  $x$  and  $(X_{(f|_F)})$  is the symplectic gradient of the restriction  $f|_F$  of  $f$  to the leaf  $F$ , with respect to the symplectic form  $\Omega_F$ .

This way, we get 5 Poisson structures on  $M_{n,k}$ .

#### Question 4

Are there any compatibility relations among these Poisson structures that can be deduced from the commutation relations of the vector fields  $\{X, Y, Z, T\}$ ?

#### Remark 4

Since  $\gamma$  is a closed 1-form without zero, a theorem of Tischler[12] asserts that  $M_{n,k}$  is fibered over  $S^1$ .

## REFERENCES

- [1] L.C. Andres, L.A. Cardero, M. Fernandez and J.J. Mencia, *Examples of four-dimensional compact locally conformal Kaehler solvmanifolds*, Geom. Dedicata 29(1989) 227– 232.
- [2] L. Auslander, L. Green, and F. Hahn, *Flows on homogeneous spaces*, Ann. of Math. Studies 53, Princeton Univ. Press, 1963.
- [3] A. Banyaga, *Sur la structure du groupe des diffeomorphismes qui preservent une forme symplectique*, Comment. Math. Helv. 53(1978) 174-227
- [4] A. Banyaga, *The structure of classical diffeomorphism Groups*. Kluwer Academic Publishers, Mathematics and its applications, Vol 400 (1997)
- [5] A. Banyaga, *Some properties of locally conformal symplectic structures*, Comment. Math. Helveticii, 77(2002) 383-398.
- [6] A. Banyaga, *On the geometry of locally conformal symplectic manifolds*, Infinite dimensional Lie Groups in Geometry and Representation Theory, A. Banyaga, J. Leslie and T. Robart , Eds. World Scientific (2002) pp 79- 91
- [7] F. Guedira and A. Lichnerowicz, *Geometrie des algebres de Lie locales de Kirillov*, J.Math. Pures et Appl. 63(1984), 407-484.
- [8] S. Haller and T. Rybickii, *On the group of diffeomorphisms preserving a locally conformal symplectic structure*, Ann. Global Anal. and Geom. 17 (1999) 475-502
- [9] H.C. Lee, *A kind of even-dimensional differential geometry and its application to exterior calculus*. Amer. J. Math. 65(1943) pp 433-438.
- [10] M. de Leon, B. Lopez, J.C. Marrero, E. Padron, *On the computation of the Lichnerowicz-Jacobi cohomology*, Jour. Geom. and Physics 44(2003) 507-522
- [11] J. Moser, *On the volume element of a manifold*, Trans. AMS 120(1965) 286-294
- [12] D. Tischler, *On fibering certain manifolds over  $S^1$* . Topology 9(1970), 153-154.

- [13] I. Vaisman, *Locally conformal symplectic manifolds*, Inter. J. Math.Sci 8(3) (1985) 521-536
- [14] I. Vaisman, *Lectures on the geometry of Poisson manifolds*, Progress in Math, Vol 118, Birkhauser, 1994
- [15] I Vaisman and S. I. Goldberg, *On compact locally conformal Kaehler manifolds with nonnegative sectional curvature*, Ann. Fac. SCi. Toulouse, 2(198), 117-123.

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